Kinematic orbits and the structure of the internal space for systems of five or more bodies

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Abstract. The internal space for a molecule, atom, or other n-body system can be conveniently parametrized by $3n-9$ kinematic angles and three kinematic invariants. For a fixed set of kinematic invariants, the kinematic angles parametrize a subspace, called a kinematic orbit, of the n-body internal space. Building on an earlier analysis of the three- and four-body problems, we derive the form of these kinematic orbits (that is, their topology) for the general n-body problem. The case $n = 5$ is studied in detail, along with the previously studied cases $n = 3, 4$.

1. Introduction

The group of kinematic rotations, called here the kinematic group, is an important set of symmetries for the n-body kinetic energy. In fact, the kinematic group, to be defined precisely below, is the largest (compact, connected) group of such symmetries acting on the n-body internal space. Not surprisingly then, the orbits (see appendix A of [1]) of the kinematic group provide a useful decomposition of the internal space. It is the purpose of this paper to analyse these orbits and to determine their topology. We have been motivated by molecular applications, but the results are quite general and could be applied to any n-body system with rotational invariance, such as atoms or nuclei.

Although many reasons exist to study kinematic rotations and their orbits, our current motivation derives from an interest in body-frame singularities and their implications for the quantum dynamics of n-body systems. In two previous papers [1, 2], body-frame singularities in the three- and four-body problems were studied explicitly. The definition of body-frame singularities, their inevitability, the flexibility one has in moving them and their importance for quantum dynamics are all discussed in [1, 2]. A detailed study of frame singularities has also been made by Pack [3]. Our earlier analysis of body-frame singularities (especially the singularities of the principal axis and related frames) involved extensive use of kinematic rotations. The present paper extends the analysis of kinematic rotations to arbitrary n and provides the basis for a future discussion of frame singularities for the general n-body problem. Although this paper concentrates on kinematic orbits in their own right, for motivational reasons, we provide a brief two-paragraph account of frame singularities, referring to [1–3] for greater detail.

An early and necessary step in many quantum n-body computations is choosing a set of body-fixed axes, that is, a body frame. The principal axis frame, in which the body-fixed axes are aligned with the principal axes, is one common choice. The body frame is a function of the

shape of the system, by which we mean the positions of the bodies relative to each other; the shape may be parametrized by $3n - 6$ internal coordinates. As has been noted previously [1–3], a body frame may fail to be a smooth function of shape, and thus there may be points in the internal, or shape, space at which it is singular. (In this paper, the internal space and shape space are synonymous.) For example, in the three-body problem, the principal axis frame is singular at all oblate symmetric tops (among other shapes), and in the four-body problem, the principal axis frame is singular at all symmetric tops (among other shapes). Body-frame singularities have important consequences for the form of the quantum wavefunction: roughly speaking, the wavefunction has singularities matching those of the body frame. An understanding of body-frame singularities is therefore critical for understanding the singularities of the n-body wavefunction.

The location of the body-frame singularities in shape space depends on the choice of body frame; by choosing different frames one can move these singularities about or possibly remove them altogether (as is essentially the case for the three-body problem [2, 3]). Thus, for many problems one can choose a frame whose singularities are outside the physically relevant region of shape space. This is true of small vibration problems (about a non-collinear equilibrium) in which the wavefunction is localized around an equilibrium shape. However, for scattering states and delocalized bound states, it becomes harder to eliminate the singularities from the region of interest. For certain regions, it becomes topologically impossible to remove them completely and one must then understand their effects.

Though the study of kinematic orbits is developed here with the ultimate intent of developing a deeper understanding of frame singularities, several other reasons motivate our work. First, since the kinematic group is the largest (compact, connected) group of symmetries (of the kinetic energy) acting on shape space, the kinematic orbits provide an important foliation of shape space with which to study the kinetic energy operator. Furthermore, this foliation suggests a convenient method of defining internal, or shape, coordinates [4–11]: three internal coordinates are chosen to be kinematic invariants (for example, the three principal moments of inertia), which label a particular kinematic orbit, and the remaining $3n - 9$ internal coordinates are chosen to be kinematic angles, which parametrize the position along the kinematic orbit. Defining these angles and properly specifying their ranges requires a clear understanding of the topology of the orbits. Finally, certain large-amplitude internal motions, such as pseudorotations, can be approximated by kinematic rotations. For such systems, it may be convenient to restrict the region of physical interest to a single kinematic orbit.

The kinematic group is commonly viewed as the set of discrete transformations between different conventions for Jacobi vectors. Here, however, we define the kinematic group to be a continuous symmetry group, namely $SO(n - 1)$, which contains these transformations. The elements of the kinematic group $SO(n - 1)$ are called kinematic rotations to distinguish them from the ordinary external $SO(3)$ rotations. (Sometimes the terminology 'democracy transformations' and 'democracy group' is used.) Kinematic rotations act (in the active sense) on the Jacobi vectors as shown in equation (2.3), from which one sees that they commute with external rotations (as shown in equation (2.1)). Therefore, kinematic rotations do indeed have a well defined action on the shape of an n-body system. It is the orbits of this action of the kinematic rotations, for arbitrary n, which we compute here.

As examples of our general analysis, we specialize to the three- and four-body problems, recovering the previously known results found in [1, 2, 12]. The kinematic orbits for the three- and four-body cases do not exhibit the full range of diversity found in the general n-body problem and are thus somewhat special. For example, in the four-body problem the kinematic orbits can be classified by whether a shape is an asymmetric top, a symmetric top or a spherical top, a classification which does not hold in the general n-body case. We therefore also specialize

to the five-body problem, for which the results are not previously known. As with the three- and four-body problems, the kinematic orbits for the five-body problem have particularly simple forms. However, there are seven classes of kinematic orbits, which is representative of the general n-body case.

The approach and methods used in this paper are geometrical in nature. We assume familiarity with the techniques of [1, 2] and some basic understanding of Lie groups, their actions on manifolds, and the quotients by such actions. Appendix A of [1] provides a useful review, as do many basic texts [13, 14].

The structure of the paper is as follows. Section 2 contains the principal derivations, in which we determine the isotropy subgroup of the kinematic action on shape space. This subgroup is related to the kinematic orbit via equation (2.5). The results of section 2 are summarized in table 1. The results for arbitrary n are discussed briefly in section 3 where we focus primarily on the collinear shapes. In section 4 we specialize the results of section 2 to the three- and four-body problems, and these results are summarized in tables 2 and 3. Similarly, in section 5, we specialize to the five-body problem. This requires substantially more work than the three- and four-body cases which causes section 5 to constitute almost half of the paper. The five-body results are summarized in table 4. Our conclusions are given in section 6. We also include an appendix containing three important theorems on the actions of Lie groups.

2. The topology of kinematic orbits for arbitrary n

In the centre-of-mass frame, the configuration of an n-body system is parametrized by $n-1$ (mass-weighted) Jacobi vectors $r_{s\alpha}$, $\alpha = 1, \ldots, n-1$. Here, the s subscript indicates that the components of $r_{s\alpha}$ are referred to a space-fixed frame. Jacobi vectors are a standard topic and we refer to the literature for more details on their definition and analysis [15, 16]. For notational convenience we also introduce the $3 \times (n-1)$ matrix F_s whose columns are the Jacobi vectors. Explicitly, $F_{si\alpha} = r_{s\alpha i}$, $i = 1, 2, 3$, $\alpha = 1, \ldots, n-1$, where $F_{si\alpha}$ and $r_{s\alpha i}$ are the components of F_s and $r_{s\alpha}$, respectively.

An ordinary rotation $\mathsf{Q} \in SO(3)$ acts on the Jacobi vectors by standard multiplication on the left

$$r_{s\alpha} \mapsto \mathsf{Q} r_{s\alpha} \tag{2.1}$$

$$\mathsf{F}_s \mapsto \mathsf{Q} \mathsf{F}_s. \tag{2.2}$$

We call such a rotation an external rotation to distinguish it from a kinematic rotation. A kinematic rotation $\mathsf{K} \in SO(n-1)$ acts by mixing up the α indices of the Jacobi vectors $r_{s\alpha}$,

$$r_{s\alpha} \mapsto \sum_\beta K_{\alpha\beta} r_{s\beta} \tag{2.3}$$

$$\mathsf{F}_s \mapsto \mathsf{F}_s \mathsf{K}^T \tag{2.4}$$

where $K_{\alpha\beta}$ denotes the components of K and T denotes the matrix transpose. Note that kinematic rotations commute with all external rotations. Thus, the kinematic group has a well defined action on the quotient of configuration space $\mathbb{R}^{3n-3}$ by the group $SO(3)$ of external rotations. This quotient is called shape space, or the internal space, and its elements are called shapes. A shape thus determines the relative positions of the bodies with respect to each other. The purpose of this section is to find the topology of the orbits of the kinematic group acting on shape space.

Considering a specific configuration F_s with shape q, the kinematic orbit Γ through q is given by (that is, diffeomorphic to)

$$\Gamma = \frac{SO(n-1)}{S} \tag{2.5}$$

where $S \subset SO(n-1)$ is the isotropy subgroup of the kinematic action at q. See theorem 1 in the appendix. The isotropy subgroup S consists of all $\mathsf{K} \in SO(n-1)$ which leave the shape q invariant. Our objective therefore is to determine S for each shape q. Now, $\mathsf{F}_s \mathsf{K}^T$ and F_s have the same shape if and only if they are related by a rotation $\mathsf{Q} \in SO(3)$. Thus, our objective is to find all $\mathsf{K} \in SO(n-1)$ such that there exists a $\mathsf{Q} \in SO(3)$ satisfying

$$\mathsf{F}_s \mathsf{K}^T = \mathsf{Q}\mathsf{F}_s. \tag{2.6}$$

We use the principal value decomposition to factor F_s into

$$\mathsf{F}_s = \mathsf{R}\mathsf{\Lambda}\mathsf{H}^T \tag{2.7}$$

where $\mathsf{R} \in SO(3)$ and $\mathsf{H} \in SO(n-1)$, and $\mathsf{\Lambda}$ is a $3 \times (n-1)$ matrix of the form

$$\mathsf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \overbrace{0 \quad 0 \quad 0}^{n-4} \quad \dots \\ 0 & \lambda_2 & 0 & 0 \quad 0 \quad 0 \quad \dots \\ 0 & 0 & \lambda_3 & 0 \quad 0 \quad 0 \quad \dots \end{bmatrix}. \tag{2.8}$$

Due to the non-uniqueness of the principal value decomposition, there is no loss of generality in assuming that $\lambda_1, \lambda_2 \geqslant 0$ and that the λ_i's are ordered as

$$\lambda_1 \geqslant \lambda_2 \geqslant |\lambda_3|. \tag{2.9}$$

Using equation (2.7), we recast the problem of satisfying equation (2.6) into the following problem: for which $\mathsf{K} \in SO(n-1)$ does there exist a $\mathsf{Q} \in SO(3)$ such that

$$\mathsf{Q}\mathsf{\Lambda}(\mathsf{H}^T \mathsf{K}\mathsf{H})^T = \mathsf{\Lambda}. \tag{2.10}$$

The above equation shows that the isotropy subgroup S of the action of the kinematic group on the shape q is conjugate to the isotropy subgroup of the action of the kinematic group on the shape of the configuration $\mathsf{\Lambda}$. Replacing S by a conjugate subgroup in equation (2.5) does not affect the resulting manifold Γ (up to a diffeomorphism). Thus, we assume without loss of generality that $\mathsf{F}_s = \mathsf{\Lambda}$, and hence we look to find all $\mathsf{K} \in SO(n-1)$ such that there exists a $\mathsf{Q} \in SO(3)$ satisfying

$$\mathsf{Q}\mathsf{\Lambda}\mathsf{K}^T = \mathsf{\Lambda}. \tag{2.11}$$

The answer depends on the rank of $\mathsf{\Lambda}$, which we denote by d. The quantity d physically represents the dimensionality of the shape q. Thus, the n-body collision has $d = 0$, linear shapes have $d = 1$, planar shapes have $d = 2$ and full three-dimensional (3D) shapes have $d = 3$.

For all values of d, $\mathsf{\Lambda}$ may be 'block-diagonalized' in the following manner:

$$\mathsf{\Lambda} = \begin{bmatrix} \mathsf{\Sigma} & 0 \\ \hline 0 & 0 \end{bmatrix} \tag{2.12}$$

where $\mathsf{\Sigma}$ is a $d \times d$ diagonal matrix and 0 represents the zero matrices of the appropriate dimensions. Equation (2.11) can only be satisfied if Q and K are also block-diagonal, having the forms

$$\mathsf{K} = \begin{bmatrix} \mathsf{A} & 0 \\ \hline 0 & \mathsf{B} \end{bmatrix} \tag{2.13}$$

$$Q = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array} \right] \tag{2.14}$$

where C and A are $d \times d$ matrices, D is a $(3-d) \times (3-d)$ matrix and B is an $(n-1-d) \times (n-1-d)$ matrix. Finding (special) orthogonal matrices K and Q satisfying equation (2.11) is then equivalent to finding orthogonal matrices A, B, C, D satisfying

$$C\Sigma A^T = \Sigma \tag{2.15}$$

$$\det A \det B = 1 \tag{2.16}$$

$$\det C \det D = 1. \tag{2.17}$$

The first equation follows from equation (2.11). The last two equations ensure that K and Q have a positive determinant.

It can be shown, due to the fact that Σ is invertible and A, C $\in O(d)$, that equation (2.15) can only be solved if C = A. This in turn allows equation (2.17) to be rewritten as

$$\det A \det D = 1. \tag{2.18}$$

Recall that D is of dimension $3 - d$. Therefore, if $d = 3$, the matrix D is completely eliminated from consideration, and equation (2.18) becomes simply

$$\det A = 1. \tag{2.19}$$

However, if $d < 3$, then there always exists an orthogonal matrix D such that $\det D = \det A$. Thus, equation (2.18) imposes no constraint whatsoever on A.

For the sake of clarity, we now summarize the problem at hand. For an arbitrary diagonal $d \times d$ matrix Σ, with non-zero eigenvalues λ_i, $i = 1, \ldots, d$ satisfying equation (2.9), we seek all matrices K $\in SO(n-1)$ given by equation (2.13), where the orthogonal $d \times d$ matrix A and orthogonal $(n-1-d) \times (n-1-d)$ matrix B satisfy

$$A\Sigma A^T = \Sigma \tag{2.20}$$

$$\det A \det B = 1 \tag{2.21}$$

$$\det A = 1 \qquad \text{(required for } d = 3 \text{ only).} \tag{2.22}$$

Note that we have eliminated all reference to the matrix Q.

To proceed we consider each of the values of $d = 0, 1, 2, 3$ separately.

Case $d = 3$

From equations (2.21) and (2.22), we see that $\det A = \det B = 1$. Thus, B is an element of $SO(n-4)$ and is independent of A. To find the allowed values of A, we consider three subcases: (i) all of the λ_i's are distinct, (ii) two of the λ_i's are equal, the third is distinct, (iii) all of the λ_i's are equal. Physically, these subcases correspond to shapes which are asymmetric tops, symmetric tops and spherical tops, respectively.

Assume subcase (i). This is perhaps the most important class of shapes since three-dimensional asymmetric tops are generic in shape space. From equation (2.20) and the fact that Σ is diagonal, A must be one of the four matrices in the group V_4, where

$$V_4 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}. \tag{2.23}$$

The group V_4 is called the viergruppe. It played a critical role in earlier analysis of the four-body problem [1, 12]; we will reproduce part of this earlier analysis in section 4. Thus, K lives in a subgroup of $SO(n-1)$ isomorphic to $V_4 \times SO(n-4)$. The group $V_4 \times SO(n-4)$ is therefore the isotropy subgroup S for three-dimensional asymmetric tops.

Assume subcase (ii). Since $\mathbf{\Sigma}$ is diagonal with two equal eigenvalues, A must be block-diagonal, with a 2×2 block which can be any element $S \in O(2)$ and a 1×1 block which must be det S to ensure the condition det A $= 1$. Thus, A lives in a subset of $SO(3)$ isomorphic to $O(2)$ and hence $O(2) \times SO(n-4)$ is the isotropy subgroup S for three-dimensional symmetric tops.

Assume subcase (iii). Since all eigenvalues of $\mathbf{\Sigma}$ are equal, $\mathbf{\Sigma}$ is proportional to the identity. Hence, A can be any element of $SO(3)$, and hence $SO(3) \times SO(n-4)$ is the isotropy subgroup S for three-dimensional spherical tops.

Case d = 2

We consider two subcases: (i) $\lambda_1 \neq \lambda_2$, (ii) $\lambda_1 = \lambda_2$. Physically, these correspond to asymmetric and symmetric tops, respectively.

Assume subcase (i). Since $\mathbf{\Sigma}$ is diagonal with two different eigenvalues, A must be one of the four matrices in the group V_4, where

$$V_4 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}. \quad (2.24)$$

This is another representation of the viergruppe in equation (2.23). For notational simplicity, we use the same symbol for both groups; it will be clear from the context which group is intended. Since B $\in O(n-3)$, the matrix K is in $V_4 \times O(n-3)$. We still must apply that remaining constraint given by equation (2.21). For this reason, we introduce the notation $V_4 \times_{\det+1} O(n-3)$ for all elements in $V_4 \times O(n-3)$ with unit determinant. Thus, $V_4 \times_{\det+1} O(n-3)$ is the isotropy subgroup S for two-dimensional asymmetric tops.

Assume subcase (ii). Since $\mathbf{\Sigma}$ is diagonal with two equal eigenvalues, $\mathbf{\Sigma}$ is proportional to the identity. Thus the matrix A can be any element of $O(2)$, and hence $O(2) \times_{\det+1} O(n-3)$ is the isotropy subgroup S for planar symmetric tops.

Case d = 1

Since A is in $O(1)$, A is either 1 or -1. From equation (2.21), A $=$ det B. Therefore, B $\in O(n-2)$ completely determines K. Thus, $O(n-2)$ is the isotropy subgroup S for linear shapes.

Case d = 0

Since the dimension of A is 0 here, there is really no matrix A to worry about. That is, K $=$ B. Thus, $SO(n-1)$ is the isotropy subgroup S for the n-body collision.

3. Comments on the general results

We summarize the results from the preceding section in table 1 and comment on a few special cases. First, as was to be expected, the kinematic orbit passing through the n-body collision is a single point $\Gamma = SO(n-1)/SO(n-1) = \{0\}$.

Table 1. Isotropy subgroups of the kinematic action on shape space.

Class	Physical description of class	Isotropy subgroup S	$\dim(\Gamma)^{a}$
3(i)	3D asymmetric top	$V_4 \times SO(n-4)$	$3n-9$
3(ii)	3D symmetric top	$O(2) \times SO(n-4)$	$3n-10$
3(iii)	3D spherical top	$SO(3) \times SO(n-4)$	$3n-12$
2(i)	Planar asymmetric top	$V_4 \times_{\det +1} O(n-3)$	$2n-5$
2(ii)	Planar symmetric top	$O(2) \times_{\det +1} O(n-3)$	$2n-6$
1	Linear shape	$O(n-2)$	$n-2$
0	n-body collision	$SO(n-1)$	0

^{a} $\Gamma =$ kinematic orbit $= SO(n-1)/S$.

A more interesting case is that of the collinear shapes. It is a well known fact that

$$\frac{SO(k+1)}{O(k)} = \mathbb{R}P^{k} \tag{3.1}$$

where $\mathbb{R}P^{k}$ is the k-dimensional real projective space ($k \geqslant 1$). The k-dimensional real projective space is the space of lines in $\mathbb{R}^{k+1}$. It may also be viewed as the k-dimensional sphere S^{k} with antipodal points identified. A quick proof of equation (3.1) can be given with the aid of theorem 1, taking

$$M = \mathbb{R}P^{k} = \left\{ \{\hat{e}, -\hat{e}\} | \hat{e} = (\hat{e}_1, \ldots, \hat{e}_{k+1}) \in S^{k} \subset \mathbb{R}^{k+1} \right\} \tag{3.2}$$

and $G = SO(k+1)$. A matrix $\mathsf{K} \in SO(k+1)$ maps $\{\hat{e}, -\hat{e}\}$ into $\{\mathsf{K}\hat{e}, -\mathsf{K}\hat{e}\}$. If $\hat{e} = (1, 0, \ldots, 0)$, then one can see that the isotropy subgroup of $\{\hat{e}, -\hat{e}\}$ is $H = O(k)$. (In fact, H is exactly the same representation of $O(k)$, $k = n-2$, discussed above for the case $d = 1$.) Since the orbit of the action of $SO(k+1)$ on $\mathbb{R}P^{k}$ is the entire space $\mathbb{R}P^{k}$, equation (3.1) follows from theorem 2 of the appendix.

Applying equation (3.1), we see that the kinematic orbit of a collinear shape is

$$\Gamma = \frac{SO(n-1)}{O(n-2)} = \mathbb{R}P^{n-2}. \tag{3.3}$$

For the four-body problem (in three dimensions) the two-fragment exit channels can be visualized as seven pairs of antipodal points on S^{2} or, equivalently, seven points on $\mathbb{R}P^{2}$ [1, 17]. Kinematic angles between these points were computed explicitly in [17]. These results were based on the understanding that $\mathbb{R}P^{2}$ is the kinematic orbit for collinear shapes in the four-body problem†. (The interest in collinear shapes stems from the fact that a two-fragment state becomes more and more collinear as the separation between the fragments increases.) The four-body work was an extension of well known results for the three-body problem (in three dimensions) in which the two-fragment exit channels can be visualized as three points on a circle $S^{1} = \mathbb{R}P^{1}$ with certain kinematic angles between them. The result presented in equation (3.3) shows that, in general, the two-fragment exit channels can be viewed as points on $\mathbb{R}P^{n-2}$, or equivalently, pairs of antipodal points on S^{n-2}. (Some quick combinatorics gives the number of points on $\mathbb{R}P^{n-2}$ to be $2^{n-1} - 1$.) Of course, this is only a topological result, and we say nothing about the values of the kinematic angles between such points.

† More precisely, the analysis of [17] is based on the fact that S^{2} is the kinematic orbit for shapes in the one-dimensional four-body problem. We ignore the one-dimensional n-body problem in this paper.

4. The three- and four-body problems

We specialize the preceding analysis to the three- and four-body problems. These cases have been studied previously [1, 2, 12]. The present analysis serves both as a check on the general results in section 2 and as practice for the five-body problem.

　　We begin with the three-body problem $n = 3$. In the analysis of section 2 we assumed for convenience that $n \geqslant 4$. However, by closely examining this analysis, one sees that the results presented in table 1 are also valid for $n = 3$, so long as one ignores the nonsensical results for the three-dimensional classes 3(i)–(iii). For classes 2(i) and (ii), the factor $O(n - 3) = O(0)$ of S is to be ignored. Thus, the isotropy subgroup of class 2(i) is the two-element group $S = \mathbb{Z}_2$ consisting of those elements of V_4 in equation (2.24) with unit determinant. Similarly, the isotropy subgroup of class 2(ii) is $SO(2)$. For class 1, the isotropy subgroup is $S = O(1) = \{+1, -1\} = \mathbb{Z}_2$. These results are summarized in table 2.

Table 2. Isotropy subgroups and kinematic orbits for the three-body problem.

Class	Physical description of class	S	$\dim(\Gamma)$	Γ[a]
2(i)	Planar asymmetric top	$\mathbb{Z}_2$	1	S^1
2(ii)	Planar symmetric top	$SO(2)$	0	$\{0\}$
1	Linear shape	$\mathbb{Z}_2$	1	S^1
0	Three-body collision	$SO(2)$	0	$\{0\}$

[a] Γ = kinematic orbit = $SO(2)/S$.

　　Since the kinematic group $SO(2)$ is particularly simple, the topology of the kinematic orbits may be presented in a more direct and illuminating form than the quotient $SO(2)/S$. These forms are recorded in the rightmost column of table 2. For the three-body problem, we can provide a convenient picture of the kinematic rotations which explains why the kinematic orbits have the topologies that they do. The three-body shape space is conveniently parametrized by three internal coordinates (w_1, w_2, w_3) with ranges $-\infty < w_1, w_2 < \infty$, $0 \leqslant w_3 < \infty$. The kinematic rotations act on shape space via standard $SO(3)$ matrices rotating about the w_3-axis. It so happens that the w_3-axis consists of the planar (non-collinear) symmetric tops as well as the three-body collision. Thus, the kinematic orbits of these shapes contain a single point, whereas the kinematic orbits of all other shapes are circles about the w_3-axis.

　　Turning to the four-body problem, we specialize the entries of table 1 for $n = 4$ and display the results in table 3. For classes 3(i)–(iii) we ignore the factor $SO(n - 4) = SO(0)$. For classes 2(i) and (ii), the factor $O(n - 3)$ reduces to $O(1) = \{+1, -1\}$. Since the choice of $+1$ or -1 in $O(1)$ is fixed by the det $= +1$ constraint, the isotropy subgroup for classes 2(i) and (ii) are V_4 and $O(2)$, respectively. An interesting observation is that the results for the four-body problem are classified solely on the basis of the symmetries of the moment-of-inertia tensor. That is, the topology of the kinematic orbit depends only on whether a shape is a spherical, symmetric, or asymmetric top. (This fact is not true for $n = 3$ or $n \geqslant 5$.)

　　In the final column of table 3, we have again represented the topologies of the kinematic orbits in a more direct and illuminating form than simply the quotient $SO(3)/S$. We have already explained in section 3, how the equality $SO(3)/O(2) = \mathbb{R}P^2$ comes about. Thus, the only orbit which requires special attention here is

$$\Gamma = \frac{SO(3)}{V_4} = \frac{SU(2)}{V_8} = \frac{S^3}{V_8}.\tag{4.1}$$

Table 3. Isotropy subgroups and kinematic orbits for the four-body problem.

Class	Physical description of class	S	$\dim(\Gamma)$	Γ^{a}
3(i)	3D asymmetric top	V_4	3	S^3/V_8
3(ii)	3D symmetric top	$O(2)$	2	$\mathbb{R}P^2$
3(iii)	3D spherical top	$SO(3)$	0	$\{0\}$
2(i)	Planar asymmetric top	V_4	3	S^3/V_8
2(ii)	Planar symmetric top	$O(2)$	2	$\mathbb{R}P^2$
1	Linear shape	$O(2)$	2	$\mathbb{R}P^2$
0	Four-body collision	$SO(3)$	0	$\{0\}$

$^{\mathrm{a}}$ Γ = kinematic orbit = $SO(3)/S$.

Here V_8 is an eight-element subgroup of $SU(2)$

$$V_8 = \{\pm\mathsf{I}, \pm\omega_1, \pm\omega_2, \pm\omega_3\} \tag{4.2}$$

where $\omega_i = -\mathrm{i}\sigma_i$, $i = 1, 2, 3$, and the σ_i's are the usual Pauli matrices. Explicitly,

$$\omega_1 = -\mathrm{i}\sigma_1 = \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix} \tag{4.3}$$

$$\omega_2 = -\mathrm{i}\sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.4}$$

$$\omega_3 = -\mathrm{i}\sigma_3 = \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}. \tag{4.5}$$

The matrices ω_i satisfy

$$\omega_1\omega_2 = -\omega_2\omega_1 = \omega_3 \tag{4.6}$$

$$\omega_2\omega_3 = -\omega_3\omega_2 = \omega_1 \tag{4.7}$$

$$\omega_3\omega_1 = -\omega_1\omega_3 = \omega_2 \tag{4.8}$$

$$\omega_i^\dagger = \omega_i^{-1} = -\omega_i. \tag{4.9}$$

These product rules show that V_8 is the quaternion group. To prove equation (4.1) we recall some basic facts about $SO(3)$. First, $SU(2)$ is the double cover of $SO(3)$, and we denote the projection by $\pi : SU(2) \to SO(3)$. The kernel of π is $\mathbb{Z}_2 = \{\mathsf{I}, -\mathsf{I}\}$ and hence $SO(3) = SU(2)/\mathbb{Z}_2$. If $\mathsf{R} = \pi(\mathsf{U})$ for some $\mathsf{U} \in SU(2)$, then R can be given explicitly by

$$R_{ij} = -\tfrac{1}{2}\,\mathrm{tr}(\omega_i\mathsf{U}\omega_j\mathsf{U}^\dagger) \tag{4.10}$$

where R_{ij}, $i, j = 1, 2, 3$, are the components of R. Using the product rules equations (4.6)–(4.9) and equation (4.10), one can verify that $\pi(V_8) = V_4$ given in equation (2.23), and hence $V_4 = V_8/\mathbb{Z}_2$. We now employ theorem 3 from the appendix, with $G = V_8$, $H = \mathbb{Z}_2 = \{\mathsf{I}, -\mathsf{I}\}$, $M = SU(2)$. (It is trivial to verify that $\mathbb{Z}_2$ is normal in V_8.) Hence, equation (A.7) yields

$$\frac{SO(3)}{V_4} = \frac{SU(2)/\mathbb{Z}_2}{V_8/\mathbb{Z}_2} = \frac{SU(2)}{V_8} = \frac{S^3}{V_8} \tag{4.11}$$

where we recall that $SU(2)$ is diffeomorphic to the three-dimensional sphere S^3.

Table 4. Isotropy subgroups and kinematic orbits for the five-body problem.

Class	Physical description of class	S	$\dim(\Gamma)$	Γ[a]
3(i)	3D asymmetric top	V_4	6	$S^3 \times (S^3/V_8)$
3(ii)	3D symmetric top	$O(2)$	5	$S^3 \times \mathbb{R}P^2$
3(iii)	3D spherical top	$SO(3)$	3	S^3
2(i)	Planar asymmetric top	$V_4 \times_{\det+1} O(2)$	5	$\mathbb{R}P^3 \times \mathbb{R}P^2$
2(ii)	Planar symmetric top	$O(2) \times_{\det+1} O(2)$	4	$(S^2 \times S^2)/\mathbb{Z}_2$
1	Linear shape	$O(3)$	3	$\mathbb{R}P^3$
0	Five-body collision	$SO(4)$	0	$\{0\}$

[a] Γ = kinematic orbit = $SO(4)/S$.

5. The five-body problem

The entries of table 1 are specialized for the five-body problem, $n = 5$, and displayed in table 4. As with the three- and four-body cases, we also represent the topology of the kinematic orbits in a more direct and illuminating form than the original quotient $\Gamma = SO(4)/S$.

Before deriving the results in table 4, we make a few observations. First, for all classes but 2(ii) we present the kinematic orbits as products of well studied two- and three-dimensional manifolds. In fact, all of these simpler manifolds already appear in the four-body problem, either as kinematic orbits or as the group manifold $SO(3) = \mathbb{R}P^3$. One obvious advantage of such a simple description of the topologies of these spaces is that it simplifies the introduction of kinematic angles or some other parametrization of the orbits. For example, in the case of a 3D asymmetric top, we may introduce six kinematic angles by taking three to be standard Euler angles on $S^3 = SU(2)$ and three to be Euler angles on S^3/V_8. The ranges of the latter three angles must be carefully restricted to account for the fact that S^3/V_8 is only one-eighth the size of S^3. A discussion of the ranges of such angles has already been given in the context of the four-body problem [11, 12, 18, 19]. Of course, the space S^3/V_8 can be parametrized in various other ways, such as the convenient coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ suggested by Reinsch [1, 20].

It is interesting to note that the kinematic orbit of a collinear shape is $\mathbb{R}P^3 = SO(3)$. A point on such an orbit (such as one of the two-fragment exit channels discussed in section 3) can therefore be identified with a rotation matrix and may, in turn, be parametrized by one of the many standard parametrizations of $SO(3)$ (Euler angles, axis–angle variables, Cayley–Klein parameters, etc).

We proceed now to derive the results of table 4.

5.1. The projection π from SU(2) × SU(2) to SO(4)

With the four-body problem, we have found it useful to work with the double cover $SU(2)$ of the kinematic group $SO(3)$. With the five-body problem, we also find it useful to work with the double cover of the kinematic group. In this case, the kinematic group is $SO(4)$ and its double cover is $SU(2) \times SU(2)$. In this section we give an explicit realization of the projection from $SU(2) \times SU(2)$ to $SO(4)$.

First, we introduce the function g which maps a complex number into a corresponding 2×2 real matrix. Explicitly,

$$g(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{5.1}$$

where a and b are real numbers. The function g is real linear and preserves multiplication.

Specifically, it is straightforward to verify the following identities:

$$g(z_1 + az_2) = g(z_1) + ag(z_2) \tag{5.2}$$

$$g(z_1 z_2) = g(z_1)g(z_2) \tag{5.3}$$

$$g(z_1^*) = g(z_1)^T \tag{5.4}$$

$$\operatorname{tr} g(z_1) = z_1 + z_1^* = 2\operatorname{Re} z_1 \tag{5.5}$$

$$\det g(z_1) = |z_1|^2 \tag{5.6}$$

$$\text{if } z_1 \neq 0 \text{ then } \quad g(z_1^{-1}) = g(z_1)^{-1} \tag{5.7}$$

where z_1 and z_2 are complex, z_1^* is the complex conjugate of z_1, and a is real. We define g acting on a $k \times k$ complex matrix to be the $2k \times 2k$ real matrix given by

$$g\left(\begin{bmatrix} a_{11} + ib_{11} & a_{12} + ib_{12} & \ldots \\ a_{21} + ib_{21} & a_{22} + ib_{22} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}\right) = \begin{bmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} & \ldots \\ b_{11} & a_{11} & b_{12} & a_{12} & \ldots \\ a_{21} & -b_{21} & a_{22} & -b_{22} & \ldots \\ b_{21} & a_{21} & b_{22} & a_{22} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.8}$$

The following identities are analogous to equations (5.2)–(5.7):

$$g(M + aN) = g(M) + ag(N) \tag{5.9}$$

$$g(MN) = g(M)g(N) \tag{5.10}$$

$$g(M^\dagger) = g(M)^T \tag{5.11}$$

$$\operatorname{tr} g(M) = \operatorname{tr} M + (\operatorname{tr} M)^* = 2\operatorname{Re}(\operatorname{tr} M) \tag{5.12}$$

$$\det g(M) = |\det M|^2 \tag{5.13}$$

$$\text{if M is invertible then} \quad g(M^{-1}) = g(M)^{-1} \tag{5.14}$$

where M and N are square complex matrices, $M^\dagger$ is the Hermitian conjugate of M, and a is a real number. Except for equation (5.13), these identities are relatively straightforward to prove. To prove equation (5.13), we first assume that M is normal and invertible, which allows us to write $M = \exp X$ for some matrix X. Then,

$$\det g(M) = \det[g(\exp X)] = \det[\exp g(X)] = \exp[\operatorname{tr} g(X)] = \exp[\operatorname{tr} X + (\operatorname{tr} X)^*]$$
$$= \det(\exp X)[\det(\exp X)]^* = |\det M|^2 \tag{5.15}$$

where we have used equations (5.12) and (5.21) (which appears below) as well as the fact that $\det \exp = \exp \operatorname{tr}$. We now consider an arbitrary (possibly non-normal) invertible matrix M and note that it may be written as a product of normal matrices (using, for example, polar or principal value decompositions). Using this fact and equation (5.10), we observe that equation (5.13) holds for M as well. Having shown that equation (5.13) is valid for all invertible matrices, analytic continuation shows that it is valid for all matrices.

We now define π from $SU(2) \times SU(2)$ to $SO(4)$ by

$$\pi(U_1, U_2) = g(U_1)Pg(U_2)P^T \tag{5.16}$$

where

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \in O(4) \tag{5.17}$$

and where $U_1, U_2 \in SU(2)$. From equations (5.11)–(5.14), we observe that $g(U_1)$ and $g(U_2)$ are in $SO(4)$. Since P is in $O(4)$ we verify that $\pi(U_1, U_2)$ is in $SO(4)$. To verify that π is a group homomorphism we must verify the following equation:

$$\pi(U_1 V_1^{-1}, U_2 V_2^{-1}) = \pi(U_1, U_2)\pi(V_1, V_2)^{-1} \tag{5.18}$$

where $U_1, U_2, V_1, V_2 \in SU(2)$ are arbitrary. To prove equation (5.18), we first hypothesize that

$$g(U)\left[Pg(V)P^T\right] = \left[Pg(V)P^T\right]g(U) \tag{5.19}$$

where $U, V \in SU(2)$ are arbitrary. We postpone the proof of equation (5.19) temporarily in order to show how it is used to prove equation (5.18). To this end, we have

$$
\begin{aligned}
\pi(U_1 V_1^{-1}, U_2 V_2^{-1}) &= g(U_1 V_1^{-1})Pg(U_2 V_2^{-1})P^T = g(U_1)g(V_1)^{-1}Pg(U_2)g(V_2)^{-1}P^T \\
&= g(U_1)g(V_1)^{-1}\left[Pg(U_2)P^T\right]\left[Pg(V_2)^{-1}P^T\right] \\
&= \left[g(U_1)Pg(U_2)P^T\right]\left[Pg(V_2)^{-1}P^T g(V_1)^{-1}\right] \\
&= \pi(U_1, U_2)\pi(V_1, V_2)^{-1}
\end{aligned}
\tag{5.20}
$$

where the first equality follows from the definition (5.16), the second from equations (5.10) and (5.14), the third from inserting $P^T P = I$, the fourth from equation (5.19) and the final equality again from equation (5.16).

We return now to prove equation (5.19). We find it convenient to work with the Lie algebras $su(2)$ and $so(4)$ of $SU(2)$ and $SO(4)$, respectively. We see from equation (5.11) that if $X \in su(2)$, that is X is a 2×2 anti-Hermitian matrix, then $g(X) \in so(4)$, that is, $g(X)$ is a 4×4 antisymmetric real matrix. Furthermore,

$$g(\exp X) = \exp g(X) \tag{5.21}$$
$$P \exp[g(X)]P^T = \exp[Pg(X)P^T] \tag{5.22}$$

where equation (5.21) follows from equations (5.9) and (5.10). Taking $U = \exp X$, $V = \exp Y$, equations (5.21) and (5.22) allow equation (5.19) to be re-expressed as

$$\exp[g(X)]\exp[Pg(Y)P^T] = \exp[Pg(Y)P^T]\exp[g(X)]. \tag{5.23}$$

The above equation is valid so long as

$$[g(X), Pg(Y)P^T] = 0 \tag{5.24}$$

for arbitrary $X, Y \in su(2)$, where $[\,,\,]$ is the matrix commutator.

We prove equation (5.24) by using a basis of $su(2)$, which we choose to be the matrices ω_i, $i = 1, 2, 3$, given in equations (4.3)–(4.5). From the product rules equations (4.6)–(4.8), this basis satisfies the Lie algebra relations

$$[\omega_i, \omega_j] = 2\sum_k \epsilon_{ijk}\omega_k. \tag{5.25}$$

It is straightforward to compute the matrices $J_i = g(\omega_i)$ and $L_i = Pg(\omega_i)P^T$,

$$J_1 = g(\omega_1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{5.26}$$

$$J_2 = g(\omega_2) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{5.27}$$

$$J_3 = g(\omega_3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.28}$$

$$L_1 = Pg(\omega_1)P^T = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{5.29}$$

$$L_2 = Pg(\omega_2)P^T = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{5.30}$$

$$L_3 = Pg(\omega_3)P^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{5.31}$$

The matrices J_i, $i = 1, 2, 3$ and L_i, $i = 1, 2, 3$ are a basis of the Lie algebra $so(4)$ and it is straightforward to show that they satisfy the Lie algebra relations

$$[J_i, J_j] = 2 \sum_k \epsilon_{ijk} J_k \tag{5.32}$$

$$[L_i, L_j] = 2 \sum_k \epsilon_{ijk} L_k \tag{5.33}$$

$$[J_i, L_j] = 0. \tag{5.34}$$

The above equations exhibit the well known fact that $so(4) = su(2) \oplus su(2)$. Since the matrices J_i and L_i span the space of matrices of the form $g(X)$ and $Pg(Y)P^T$, respectively ($X, Y \in su(2)$), equation (5.34) proves equation (5.24), from which follows equations (5.23), (5.19) and (5.18). We have thus shown π to be a group homomorphism.

The mapping π has several important properties which we will use later. First, it follows directly from the definition (5.16) that for arbitrary group elements $(U_1, U_2) \in SU(2) \times SU(2)$,

$$\pi(U_1, -U_2) = \pi(-U_1, U_2) = -\pi(U_1, U_2) \tag{5.35}$$

$$\pi(-U_1, -U_2) = \pi(U_1, U_2). \tag{5.36}$$

Second, in light of equation (5.34), the definition of π can be conveniently re-expressed using the Lie algebra,

$$\pi(\exp X_1, \exp X_2) = \exp[g(X_1) + Pg(X_2)P^T]. \tag{5.37}$$

Since the matrices J_i and L_i form a basis of the Lie algebra $so(4)$, the above equation shows that π is surjective. However, π is obviously not injective since $\pi(I, I) = \pi(-I, -I) = I$. In fact, (I, I) and $(-I, -I)$ are the only two elements of $SU(2) \times SU(2)$ which map to $I \in SO(4)$. To prove this we consider two arbitrary elements $U_1, U_2 \in SU(2)$, expressed in axis-angle form as

$$U_1 = \cos\theta_1\, I - \sin\theta_1\, \hat{\boldsymbol{n}}_1 \cdot \boldsymbol{\omega} \tag{5.38}$$

$$U_2 = \cos\theta_2\, I - \sin\theta_2\, \hat{\boldsymbol{n}}_2 \cdot \boldsymbol{\omega} \tag{5.39}$$

where θ_1, θ_2 are rotation angles, $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2$ are rotation axes and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. Then,

$$\begin{aligned}\pi(U_1, U_2) &= (\cos\theta_1 I - \sin\theta_1\hat{\boldsymbol{n}}_1 \cdot \boldsymbol{J})(\cos\theta_2 I - \sin\theta_2\hat{\boldsymbol{n}}_2 \cdot \boldsymbol{L}) \\ &= \cos\theta_1 \cos\theta_2 I - \sin\theta_1 \cos\theta_2(\hat{\boldsymbol{n}}_1 \cdot \boldsymbol{J}) \\ &\quad - \cos\theta_1 \sin\theta_2(\hat{\boldsymbol{n}}_2 \cdot \boldsymbol{L}) + \sin\theta_1 \sin\theta_2(\hat{\boldsymbol{n}}_1 \cdot \boldsymbol{J})(\hat{\boldsymbol{n}}_2 \cdot \boldsymbol{L}) \end{aligned} \tag{5.40}$$

where $\boldsymbol{J} = (J_1, J_2, J_3)$ and $\boldsymbol{L} = (L_1, L_2, L_3)$. It can easily be verified that $\{I, J_i, L_i, J_iL_j\}$ forms a basis of all 4×4 real matrices. Thus, if $\pi(U_1, U_2) = I$ then

$$\cos\theta_1 \cos\theta_2 = 1 \tag{5.41}$$

$$\sin\theta_1 \cos\theta_2 = 0 \tag{5.42}$$

$$\cos\theta_1 \sin\theta_2 = 0 \tag{5.43}$$

$$\sin\theta_1 \sin\theta_2 = 0 \tag{5.44}$$

which only occurs if $\theta_1 = \theta_2 = 0$ or $\theta_1 = \theta_2 = \pi$, corresponding to $U_1 = U_2 = I$ or $U_1 = U_2 = -I$, respectively.

In summary, we have proved that π given by equation (5.16) is a two-to-one surjective group homomorphism from $SU(2) \times SU(2)$ to $SO(4)$.

5.2. The double covers of the isotropy subgroups

For each isotropy subgroup S, we determine its double cover $\hat{S} = \pi^{-1}(S)$. That is we must find the two elements of $SU(2) \times SU(2)$ which map to each element of S. In light of equation (5.36), these two elements are related by a minus sign, that is, $(U_1, U_2) \in \hat{S}$ and $(-U_1, -U_2) \in \hat{S}$ map to the same element in S. Thus, the problem of determining $\hat{S}$ reduces to finding only one element in $SU(2) \times SU(2)$ which maps to each element in S.

Since $\mathbb{Z}_2 = \{(I, I), -(I, I)\}$ is obviously normal in $\hat{S}$ and $(SU(2) \times SU(2))/\mathbb{Z}_2 = SO(4)$ and $\hat{S}/\mathbb{Z}_2 = S$, we apply theorem 3 to find

$$\Gamma = \frac{SO(4)}{S} = \frac{(SU(2) \times SU(2))/\mathbb{Z}_2}{\hat{S}/\mathbb{Z}_2} = \frac{SU(2) \times SU(2)}{\hat{S}}. \tag{5.45}$$

We will use this result extensively to determine the topology of the kinematic orbits. We analyse each class in table 4 separately.

5.2.1. The class 3(i) of 3D asymmetric tops.
Considering the analysis of section 2, an element K of S depends on the two matrices A and B as shown in equation (2.13). Considering class 3(i) for $n = 5$, the matrix B is simply the 1×1 matrix $B = 1$. The matrix A must belong to the

group V_4 given by equation (2.23). Thus, the group S contains the following matrices:

$$S = \{I, E_1, E_2, E_3\} \tag{5.46}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.47}$$

$$E_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.48}$$

$$E_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.49}$$

From equation (5.16) we see that $\pi(\omega_i, \omega_i) = J_i L_i$, where J_i and L_i are given by equations (5.26)–(5.31). By direct matrix multiplication, we find

$$\pi(\omega_i, \omega_i) = J_i L_i = E_i. \tag{5.50}$$

Thus, having found one element in $SU(2) \times SU(2)$ which maps to each element of S, the double cover $\hat{S}$ is the eight-element group

$$\hat{S} = \{\pm(I, I), \pm(\omega_1, \omega_1), \pm(\omega_2, \omega_2), \pm(\omega_3, \omega_3)\} = \tilde{V}_8. \tag{5.51}$$

In the above, the tilde over V_8 has a technical meaning which we now define. If H is an arbitrary subgroup of $SU(2)$, then $\tilde{H}$ is an isomorphic subgroup of $SU(2) \times SU(2)$ as shown in equation (A.1). Thus, $\hat{S}$ is isomorphic to the quaternion group in equation (4.2). Since $\hat{S}$ has the form of $\tilde{H}$ in equation (A.1), we apply theorem 2 from the appendix to find

$$\Gamma = \frac{SU(2) \times SU(2)}{\hat{S}} = SU(2) \times \frac{SU(2)}{V_8} = S^3 \times \frac{S^3}{V_8}. \tag{5.52}$$

5.2.2. The class 3(ii) of 3D symmetric tops. For class 3(ii), B is again the 1×1 matrix $B = 1$. The matrix A consists of a 2×2 block $S \in O(2)$ and a 1×1 block $\det S$. Combining these results into the single matrix K, we see that S consists of

$$S = \left\{ \left[\begin{array}{c|cc} S & & 0 \\ \hline 0 & \det S & 0 \\ & 0 & 1 \end{array} \right] \;\middle|\; S \in O(2) \right\} = \{I, E_1\} \left\{ \left[\begin{array}{c|c} S & 0 \\ \hline 0 & I \end{array} \right] \;\middle|\; S \in SO(2) \right\} \tag{5.53}$$

where in the second equality we have factored S into the product of two groups. Having encountered the elements of the first factor earlier, we recall that $\pi(I, I) = I$ and (from

equation (5.50)) that $\pi(\omega_1, \omega_1) = \mathsf{E}_1$. Considering the second factor, we note

$$\pi(\exp(\theta\omega_3), \exp(\theta\omega_3)) = \exp(\theta(\mathsf{J}_3 + \mathsf{L}_3)) = \exp\left(2\theta\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline & 0 & 0 \end{array}\right]\right)$$

$$= \left[\begin{array}{cc|c} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & \\ \hline & 0 & \mathsf{I} \end{array}\right] \tag{5.54}$$

where we have used equations (5.37), (5.28) and (5.31). Equation (5.54) shows that the double cover of the second factor in equation (5.53) is the group

$$\tilde{A} = \{(\mathsf{U}, \mathsf{U})|\mathsf{U} \in A\} \tag{5.55}$$

where A is the group

$$A = \{\exp(\theta\omega_3)|0 \leqslant \theta < 2\pi\}. \tag{5.56}$$

Note that $A = U(1) \subset SU(2)$. The double cover of S is therefore

$$\hat{S} = \{(\mathsf{I}, \mathsf{I}), (\omega_1, \omega_1)\}\tilde{A} = \{(\mathsf{U}, \mathsf{U})|\mathsf{U} \in B\} = \tilde{B} \tag{5.57}$$

where B is the group

$$B = \{\mathsf{I}, \omega_1\}A. \tag{5.58}$$

A priori, it is perhaps not obvious that B is actually a group. To verify that B is indeed closed under multiplication and inverses, the following identity is useful:

$$\omega_1 \exp(\theta\omega_3)\omega_1^\dagger = \exp(\theta\omega_1\omega_3\omega_1^\dagger) = \exp(-\theta\omega_3) \tag{5.59}$$

which is derived from equations (4.6)–(4.9). When forming products and inverses of the elements of B, equation (5.59) allows any ω_1 factors to be shifted to the left so that the final result again has the form displayed in equation (5.58).

Since $\hat{S}$ has the form of equation (A.1), we apply theorem 2 to find

$$\Gamma = \frac{SU(2) \times SU(2)}{\hat{S}} = \frac{SU(2) \times SU(2)}{\tilde{B}} = SU(2) \times \frac{SU(2)}{B}. \tag{5.60}$$

The quotient $SU(2)/B$ is diffeomorphic to $\mathbb{R}P^2$. To prove this, we first consider the quotient $SU(2)/A = SU(2)/U(1)$. It is well known that $SU(2)/U(1) = S^2$. One way of seeing this fact is to consider the action of $SU(2)$ on $\mathbb{R}^3$ via the 3×3 orthogonal matrices given in equation (4.10). The orbit of $SU(2)$ acting on $\hat{z} \in \mathbb{R}^3$ is clearly the sphere S^2. The isotropy subgroup of the vector $\hat{z}$ is the group A of $U(1)$ rotations about the $\hat{z}$-axis. Theorem 1 thus gives the desired result

$$\frac{SU(2)}{A} = S^2. \tag{5.61}$$

Furthermore, we note the following explicit identification between a right coset $[\mathsf{U}] = A\mathsf{U} \in SU(2)/A$ and a unit vector $\hat{n} \in S^2$ (denoting a coset with bold square brackets),

$$[\mathsf{U}] \leftrightarrow \hat{n} = \mathsf{R}^T\hat{z} = -\tfrac{1}{2}\mathrm{tr}(\omega\mathsf{U}^\dagger\omega_3\mathsf{U}) \tag{5.62}$$

where R is given by equation (4.10) and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. We have placed the transpose on R in order that $\hat{\boldsymbol{n}}$ be well defined for right cosets; observe that the right-hand side of equation (5.62) is invariant under $\mathsf{U} \mapsto \exp(\theta\omega_3)\mathsf{U}$.

Having computed $SU(2)/A$, we apply theorem 3 to compute $SU(2)/B$. With regard to the notation of the theorem, we take $G = B$, $H = A$ and $M = SU(2)$. We first must verify that A is normal in B. Proving this fact reduces to showing that $\omega_1 \exp(\theta\omega_3)\omega_1^\dagger$ is in A for an arbitrary $\exp(\theta\omega_3) \in A$. This fact, in turn, follows immediately from equation (5.59). Thus, A is normal in B and B/A is a well defined group isomorphic to $\mathbb{Z}_2$. According to theorem 3, the non-identity element $[\omega_1] \in B/A$ acts on $[\mathsf{U}] \in SU(2)/A$ by $[\omega_1][\mathsf{U}] = [\omega_1\mathsf{U}]$. Identifying $[\mathsf{U}]$ with $\hat{\boldsymbol{n}}$, the action of $[\omega_1]$ on $\hat{\boldsymbol{n}}$ is

$$[\omega_1]\hat{\boldsymbol{n}} = -\tfrac{1}{2}\,\mathrm{tr}[\boldsymbol{\omega}(\omega_1\mathsf{U})^\dagger\omega_3(\omega_1\mathsf{U})] = \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\omega}\mathsf{U}^\dagger\omega_3\mathsf{U}) = -\hat{\boldsymbol{n}} \tag{5.63}$$

which follows from equations (4.6)–(4.9). Thus, the quotient $S^2/\mathbb{Z}_2$ is $\mathbb{R}P^2$, and by theorem 3 we have the following identifications:

$$\frac{SU(2)}{B} = \frac{SU(2)/A}{B/A} = \frac{S^2}{\mathbb{Z}_2} = \mathbb{R}P^2. \tag{5.64}$$

Recalling equation (5.60), we find that

$$\Gamma = S^3 \times \mathbb{R}P^2. \tag{5.65}$$

5.2.3. The class 3(iii) of 3D spherical tops. For class 3(iii), B is again the 1×1 matrix $\mathsf{B} = 1$. The matrix A can be any matrix in $SO(3)$. Thus, the group S is

$$S = \left\{ \left[\begin{array}{c|c} \mathsf{A} & 0 \\ \hline 0 & 1 \end{array} \right] \;\middle|\; \mathsf{A} \in SO(3) \right\}. \tag{5.66}$$

To find the double cover of S, we consider an arbitrary matrix $\mathsf{U} \in SU(2)$ expressed as $\mathsf{U} = \exp(\boldsymbol{n} \cdot \boldsymbol{\omega})$ for some vector $\boldsymbol{n} = (n_1, n_2, n_3)$. Then we find from equation (5.37) and equations (5.26)–(5.31) that

$$\pi(\mathsf{U}, \mathsf{U}) = \pi(\exp(\boldsymbol{n} \cdot \boldsymbol{\omega}), \exp(\boldsymbol{n} \cdot \boldsymbol{\omega})) = \exp[\boldsymbol{n} \cdot (\boldsymbol{J} + \boldsymbol{L})]. \tag{5.67}$$

Furthermore, from equations (5.26)–(5.31), we see that

$$\mathsf{J}_1 + \mathsf{L}_1 = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.68}$$

$$\mathsf{J}_2 + \mathsf{L}_2 = 2 \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.69}$$

$$\mathsf{J}_3 + \mathsf{L}_3 = 2 \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.70}$$

Thus, the matrices $\mathsf{J}_i + \mathsf{L}_i$ generate S, and from equation (5.67) the double cover of S is

$$\hat{S} = \{(\mathsf{U}, \mathsf{U}) | \mathsf{U} \in SU(2)\} = \tilde{E}. \tag{5.71}$$

Applying theorem 2, we have

$$\Gamma = \frac{SU(2) \times SU(2)}{\hat{S}} = \frac{SU(2) \times SU(2)}{\tilde{E}} = SU(2) \times \frac{SU(2)}{SU(2)} = S^3. \tag{5.72}$$

5.2.4. The class 2(i) of planar asymmetric tops. For class 2(i), the matrices A and B are, respectively, in V_4 (as shown in equation (2.24)) and $O(2)$. These matrices must further satisfy det AB $= 1$. For convenience, we switch the positions of A and B in equation (2.13). That is, we place A in the lower right-hand block and B in the upper left-hand block. This switch is equivalent to conjugating by a permutation and hence does not effect the topology of the quotient $SO(4)/S$. With this modification, the isotropy subgroup is

$$S = \left\{ \left[\begin{array}{c|c} \mathsf{B} & 0 \\ \hline 0 & \mathsf{A} \end{array} \right] \middle| \mathsf{B} \in O(2), \mathsf{A} \in V_4, \det \mathsf{AB} = 1 \right\}$$

$$= \{\mathsf{I}, \mathsf{E}_1, -\mathsf{E}_2, -\mathsf{E}_3\} \left\{ \left[\begin{array}{c|c} \mathsf{B} & 0 \\ \hline 0 & \mathsf{I} \end{array} \right] \middle| \mathsf{B} \in SO(2) \right\} \tag{5.73}$$

where we have again factored S into the product of two groups. Considering the elements in the first factor, we may combine equations (5.35) and (5.50) to produce

$$\pi(\omega_1, \omega_1) = \mathsf{E}_1 \tag{5.74}$$

$$\pi(\omega_2, -\omega_2) = -\mathsf{E}_2 \tag{5.75}$$

$$\pi(-\omega_3, \omega_3) = -\mathsf{E}_3. \tag{5.76}$$

The second factor of equation (5.73) is identical to the second factor of equation (5.53), and hence the double cover of the second factor is $\tilde{A}$ given by equation (5.55). Thus, the double cover of S is

$$\hat{S} = \{(\mathsf{I}, \mathsf{I}), (\omega_1, \omega_1), (\omega_2, -\omega_2), (-\omega_3, \omega_3)\} \tilde{A}$$

$$= \{(\mathsf{I}, \mathsf{I}), (\omega_2, -\omega_2)\}\{(\mathsf{I}, \mathsf{I}), (\omega_1, \omega_1)\}\tilde{A} = \{(\mathsf{I}, \mathsf{I}), (\omega_2, -\omega_2)\}\tilde{B} \tag{5.77}$$

where $\tilde{B}$ is given in equation (5.57).

We apply theorem 3 to determine the topology of the kinematic orbit, taking $G = \hat{S}$, $H = \tilde{B}$ and $M = SU(2) \times SU(2)$. Using equations (4.6)–(4.9), it is straightforward to verify that $\tilde{B}$ is normal in $\hat{S}$, and hence $\hat{S}/\tilde{B}$ is a well defined group isomorphic to $\mathbb{Z}_2$. The action of the non-identity element $[\omega_2, -\omega_2] \in \hat{S}/\tilde{B}$ on $[\mathsf{U}_1, \mathsf{U}_2] \in (SU(2) \times SU(2))/\tilde{B}$ is

$$[\omega_2, -\omega_2][\mathsf{U}_1, \mathsf{U}_2] = [\omega_2 \mathsf{U}_1, -\omega_2 \mathsf{U}_2]. \tag{5.78}$$

Using theorem 2, we previously showed that $(SU(2) \times SU(2))/\tilde{B}$ is diffeomorphic to $SU(2) \times (SU(2)/B)$. The diffeomorphism is given by equation (A.4). Using this diffeomorphism we find that the action of $[\omega_2, -\omega_2] \in \hat{S}/\tilde{B}$ on $(\mathsf{U}_1, [\mathsf{U}_2]) \in SU(2) \times (SU(2)/B)$ is

$$[\omega_2, -\omega_2](\mathsf{U}_1, [\mathsf{U}_2]) = (-\mathsf{U}_1, [-\omega_2 \mathsf{U}_2]) = (-\mathsf{U}_1, [\mathsf{U}_2]) \tag{5.79}$$

where the last equality follows from the fact that $-\omega_2 = \omega_1 \omega_3 \in B$. (Be careful not to confuse the bold square bracket notation [,] used for cosets of $SU(2) \times SU(2)$ with the (non-bold) square bracket notation used for the matrix commutator.) From equation (5.79) we see that the quotient of $SU(2) \times (SU(2)/B)$ by $\hat{S}/\tilde{B}$ is $\mathbb{R}P^3 \times (SU(2)/B)$. Applying theorem 3 and recalling equation (5.64), we find

$$\Gamma = \frac{SU(2) \times SU(2)}{\hat{S}} = \frac{(SU(2) \times SU(2))/\tilde{B}}{\hat{S}/\tilde{B}} = \frac{SU(2) \times (SU(2)/B)}{\hat{S}/\tilde{B}} = \mathbb{R}P^3 \times \mathbb{R}P^2. \tag{5.80}$$

5.2.5. The class 2(ii) of planar symmetric tops. For class 2(ii), the matrices A and B are both in $O(2)$ and satisfy det $AB = 1$. Thus, the isotropy subgroup is

$$S = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right] \middle| A, B \in O(2), \det AB = 1 \right\}$$

$$= \{I, E_1\} \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array} \right] \middle| A \in SO(2) \right\} \left\{ \left[\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array} \right] \middle| B \in SO(2) \right\} \quad (5.81)$$

where we have factored S into three factors. The first two factors multiply to give the group S in equation (5.53). Thus, the double cover of the first two factors is the group $\tilde{B}$ in equation (5.57). Considering the last factor of equation (5.81), we note

$$\pi\left(\exp(\theta\omega_3), \exp(-\theta\omega_3) \right) = \exp\left(\theta(J_3 - L_3) \right) = \exp\left(2\theta \left[\begin{array}{c|c} \begin{array}{cc} 0 & \\ & 0 \end{array} & 0 \\ \hline 0 & \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \end{array} \right] \right)$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \begin{array}{cc} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{array} \end{array} \right] \quad (5.82)$$

where we used equations (5.37), (5.28) and (5.31). Thus, the double cover of the third factor in equation (5.81) is the group

$$C = \{(U, U^\dagger) | U \in A\} \quad (5.83)$$

where A is the group defined in equation (5.56). Therefore, the double cover of S is given by the product of $\tilde{B}$ and C,

$$\hat{S} = \tilde{B}C = \{(I, I), (\omega_1, \omega_1)\} \tilde{A}C = \{(I, I), (\omega_1, \omega_1)\} D \quad (5.84)$$

where we have used equation (5.57) and where

$$D = \tilde{A}C = \{(U_1, U_2) | U_1, U_2 \in A\} = U(1) \times U(1). \quad (5.85)$$

We apply theorem 3, with $G = \hat{S}$, $H = D$ and $M = SU(2) \times SU(2)$, to determine the topology of the kinematic orbit. Using equation (5.59), it is straightforward to verify that D is normal in $\hat{S}$, and hence $\hat{S}/D$ is a well defined group isomorphic to $\mathbb{Z}_2$. Since $D = U(1) \times U(1)$, we find

$$\frac{SU(2) \times SU(2)}{D} = \frac{SU(2) \times SU(2)}{U(1) \times U(1)} = \frac{SU(2)}{A} \times \frac{SU(2)}{A} = S^2 \times S^2 \quad (5.86)$$

where we have used equation (5.61). From equation (5.63), the action of the non-trivial element $[\omega_1, \omega_1] \in \hat{S}/D$ on $(\hat{n}_1, \hat{n}_2) \in S^2 \times S^2$ is shown to be

$$[\omega_1, \omega_1](\hat{n}_1, \hat{n}_2) = (-\hat{n}_1, -\hat{n}_2). \quad (5.87)$$

With this understanding of the action of $\mathbb{Z}_2$ on $S^2 \times S^2$, we have

$$\Gamma = \frac{SU(2) \times SU(2)}{\hat{S}} = \frac{(SU(2) \times SU(2))/D}{\hat{S}/D} = \frac{S^2 \times S^2}{\mathbb{Z}_2}. \quad (5.88)$$

5.2.6. The class 1 of collinear shapes.   In section 3 we showed that for collinear shapes $\Gamma = SO(n-1)/O(n-2) = \mathbb{R}P^{n-2}$. For $n = 5$ this yields $\mathbb{R}P^3$ and no more need be said. However, for completeness and analogy with the preceding cases, we show here how this result also follows from equation (5.45).

Matrix B can be any matrix in $O(3)$ and matrix A is the 1×1 matrix A = det B. As in the analysis of class 2(i), we switch the positions of the blocks in K containing A and B. Specifically, the isotropy subgroup is

$$
S = \left\{ \left[\begin{array}{c|c} B & 0 \\ \hline 0 & \det B \end{array} \right] \middle| \; B \in O(3) \right\}
$$

$$
= \{I, -I\} \left\{ \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array} \right] \middle| \; B \in SO(3) \right\}. \tag{5.89}
$$

Concerning the first factor, equation (5.35) shows

$$
\pi(-I, I) = -I. \tag{5.90}
$$

The second factor is the same as the group S given in equation (5.66). Thus, the double cover of the second factor is the group $\tilde{E}$ given in equation (5.71) and the double cover of S is

$$
\hat{S} = \{(I, I), (-I, I)\}\tilde{E}. \tag{5.91}
$$

We apply theorem 3 with $G = \hat{S}$, $H = \tilde{E}$, and $M = SU(2) \times SU(2)$. It is trivial to show that $\tilde{E}$ is normal in $\hat{S}$ and hence $\hat{S}/\tilde{E}$ is a well defined group isomorphic to $\mathbb{Z}_2$. Recall from equation (5.72) that $(SU(2) \times SU(2))/\tilde{E} = SU(2)$. The non-trivial element $[-I, I] \in \hat{S}/\tilde{E}$ maps $[U_1, U_2] \in (SU(2) \times SU(2))/\tilde{E}$ into $[-U_1, U_2]$. Using the diffeomorphism $f : (SU(2) \times SU(2))/\tilde{E} \to SU(2)$ of equation (A.4), this results in the following action on $U \in SU(2)$:

$$
[-I, I]U = -U. \tag{5.92}
$$

Thus, we find

$$
\Gamma = \frac{SU(2) \times SU(2)}{\hat{S}} = \frac{(SU(2) \times SU(2))/\tilde{E}}{\hat{S}/\tilde{E}} = \frac{SU(2)}{\mathbb{Z}_2} = \mathbb{R}P^3. \tag{5.93}
$$

6. Conclusions

For the general n-body problem, we have expressed a kinematic orbit as the quotient of the kinematic group by the isotropy subgroup of the shape in question. We have computed these isotropy subgroups explicitly. For the three-, four- and five-body cases, we have represented the kinematic orbits in terms of simple well studied spaces of low dimension. We have also showed that the kinematic orbit of a collinear shape is $\mathbb{R}P^{n-2}$ for any n.

The natural next step for us to take is an analysis of body-frame singularities for $n \geqslant 5$. We envision such an analysis beginning, as in the case of the three- and four-body analysis [1, 2], with a detailed study of the principal axis frame and its singularities. As in the previous analysis, this would amount to finding the fundamental group of the asymmetric top region of

shape space and then relating the paths (or more precisely the equivalence classes of paths) in this group to the jumps in the principal axis frame.

In the three- and four-body problems, one can find a frame related to the principal axis frame which has a smaller set of frame singularities. In particular, the frame jumps can be completely eliminated. A natural question is whether such a frame exists for $n \geqslant 5$. Extending this line of inquiry, another natural question is which frames have the smallest set of singularities and what constraints are placed on one's ability to move these singularities around. We believe that the study of frames restricted to the kinematic orbits may shed some light on these issues. For example, it would be useful to know, in the language of fibre bundles, whether the $SO(3)$ bundles defined over the kinematic orbits are trivial or not.

Acknowledgments

Appendix. Theorems on Lie group quotients

We present three theorems regarding the actions of Lie groups on manifolds and the corresponding quotient spaces. These results provide a rigorous mathematical foundation for many of the steps presented in the bulk of the paper. The first result is a standard theorem and is found, for example, in Bredon ([14], p 303, corollary 1.3).

Theorem 1. *Let G be a compact Lie group acting smoothly on a smooth manifold M. Then the orbit through a point $x \in M$ is diffeomorphic to G/H where H is the isotropy subgroup of G at x. (That is, H contains all elements of G which leave x fixed.)*

The next result is useful for simplifying the descriptions of several manifolds appearing in the five-body problem. It is similar to an exercise of Bredon ([14], p 113, exercise 9).

Theorem 2. *Let G be a compact Lie group and H a Lie subgroup of G. Let $\tilde{H}$ be the following Lie subgroup of $G \times G$:*

$$\tilde{H} = \{(h, h) | h \in H\}. \tag{A.1}$$

Of course, $\tilde{H}$ is trivially isomorphic to H. Then, the smooth manifolds $(G \times G)/\tilde{H}$ and $G \times (G/H)$ are diffeomorphic.

Proof. Assuming that $(G \times G)/\tilde{H}$ and G/H are the right coset spaces, we introduce the following notation for the right cosets:

$$[g_1, g_2]_{\tilde{H}} = \tilde{H}(g_1, g_2) \in (G \times G)/\tilde{H} \qquad g_1, g_2 \in G \tag{A.2}$$

$$[g]_H = Hg \in G/H \qquad g \in G. \tag{A.3}$$

We define a function $f : (G \times G)/\tilde{H} \to G \times (G/H)$ acting on an arbitrary $[g_1, g_2]_{\tilde{H}} \in (G \times G)/\tilde{H}$ by

$$f([g_1, g_2]_{\tilde{H}}) = (g_2^{-1}g_1, [g_2]_H). \tag{A.4}$$

We assert that f is a diffeomorphism. First, we verify that f is well defined on the coset space by noting

$$f([hg_1, hg_2]_{\tilde{H}}) = ((hg_2)^{-1}(hg_1), [hg_2]_H) = (g_2^{-1}g_1, [g_2]_H) = f([g_1, g_2]_{\tilde{H}}) \tag{A.5}$$

where $h \in H$ is arbitrary. Next, it is straightforward to verify that the following function is well defined and that it is the inverse of f:

$$f^{-1}(g_1, [g_2]_H) = [g_2 g_1, g_2]_{\tilde{H}} \tag{A.6}$$

where $g_1, g_2 \in G$ are arbitrary. Since both f and f^{-1} are smooth, they are both diffeomorphisms. $\qquad\square$

The following theorem is a refinement of an exercise in Bredon ([14], p 67, exercise 1) to the case of smooth actions. We omit the straightforward proof.

Theorem 3. *Let G be a compact Lie group and H a normal Lie subgroup of G so that G/H is itself a Lie group. Let G act smoothly upon a smooth manifold M. Assume that the isotropy subgroups of this action are all conjugate to one another so that M/G and M/H are themselves smooth manifolds. Then, G/H has a well defined action on M/H given by $[g]_H[x]_H = [gx]_H$, where $[g]_H \in G/H$ and $[x]_H \in M/H$. Furthermore, the following diffeomorphism holds:*

$$\frac{M}{G} = \frac{M/H}{G/H}. \tag{A.7}$$

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